J Glob Optim (2006) 36:339–349
DOI 10.1007/s10898-006-9012-5

ORIGINAL ARTICLE

Strong vector equilibrium problems

Xunhua Gong

Received: 14 July 2004 / Accepted: 21 February 2006 / Published online: 4 July 2006
© Springer Science+Business Media B.V. 2006

Abstract In this paper, the existence of the solution for strong vector equilibrium problems is studied by using the separation theorem for convex sets. The arc-wise connectedness and the closedness of the strong solution set for vector equilibrium problems are discussed; and a necessary and sufficient condition for the strong solution is obtained.

Keywords Vector equilibrium problem · Strong solution · Arc-wise connectedness · Closedness · Necessary and sufficient condition

1 Introduction

Let X be a real Hausdorff topological vector space, A a nonempty subset of X, and $g: A \times A \to R$. Then the scalar equilibrium problem consists in finding $x_0 \in A$ such that

$$g(x_0, x) \geq 0 \quad \text{for all } x \in A.$$

This problem provides a unifying framework for some important problems such as optimization, saddle point problem, Nash equilibrium problem, fixed point problem, variational inequality, and complementarity problems and has wide applications in mathematical economics and mechanics (see Blum and Oettli [1]).

Now let Z be another real locally convex Hausdorff space, and Z^* the topological dual space of Z, and let $C \subset Z$ be a closed convex pointed cone. Let $C^* = \{z^* \in Z^* : z^*(z) \geq 0$ for all $z \in C\}$ be the dual cone of C. The cone C induces a partially ordering in Z, defined by

$$z_1 \leq z_2 \quad \text{if and only if} \quad z_2 - z_1 \in C.$$

Let int C denote the topological interior of C.

X. H. Gong (✉)
Department of Mathematics, Nanchang University, Nanchang 330047, People's Republic of China
e-mail: ncxhgong@263.net

Let $f : A \times A \to Z$ be a given mapping. Ansari et al. [2] introduced the following vector equilibrium problems: to find $x_0 \in A$ such that

$$f(x_0, x) \notin -\text{int}C \quad \text{for all } x \in A, \tag{1.1}$$

or to find $x_0 \in A$ such that

$$f(x_0, x) \geq 0 \quad \text{for all } x \in A. \tag{1.2}$$

Both problems constitute a valid extension of the scalar equilibrium problem. In case that the map f is multivalued, Ansari et al. [2] also introduced the following set-valued vector equilibrium problem: to find $x_0 \in A$ such that

$$f(x_0, x) \not\subset -\text{int}C \quad \text{for all } x \in A, \tag{1.3}$$

or to find $x_0 \in A$ such that

$$f(x_0, x) \subset C \quad \text{for all } x \in A. \tag{1.4}$$

In [2], Ansari et al. gave an existence theorem for problem (1.4).
If int $C \neq \emptyset$, and x_0 satisfies (1.1), then we call x_0 a weak efficient solution for vector equilibrium problem (for short, VEP) and call (1.1) a weak vector equilibrium problem.
If x_0 satisfies (1.2), then we call x_0 a strong solution for VEP, and call (1.2) a strong vector equilibrium problem. Denote the set of all strong solutions to the VEP by $V_S(A, f)$.

If $x_0 \in A$ satisfies

$$f(x_0, x) \notin -C\backslash\{0\} \quad \text{for all } x \in A, \tag{1.5}$$

then we call x_0 an efficient solution for VEP. Denote the set of all efficient solutions to the VEP by $V(A, f)$.

Up to now, many authors have studied the vector equilibrium problems (see [2–24]), focusing mainly on the study of the existence of weakly efficient solution for the vector equilibrium problems.

If int $C = \emptyset$, then the weak vector equilibrium problems cannot be studied. We know that many a positive cones has an empty interior. For example, in the normed spaces l^p and $L^p(\Omega)$, where $1 < p < +\infty$, the standard ordering cone has an empty interior. In this case, we can study the existence of the efficient solution and the strong solution, and the properties of those solutions set. Giannessi et al. [3] studied the properties of the efficient solutions.

When int $C = \emptyset$, in order to study the VEP, Gong [4, 5] introduced the concept of the proper efficient solutions for VEP, such as Henig efficient solution and the super efficient solution for VEP. In addition, Gong gave the scalarization results for Henig efficient solution and the super efficient solution to the VEP; discussed the existence of the efficient solution, Henig efficient solution, and the super efficient solution; and studied the connectedness of Henig efficient solutions set and the super efficient solutions set to the VEP.

We have the following relationships among the various solutions for the VEP.

A strong solution is a super efficient solution, if C is a self-allied cone (the definition of the self-allied cone can be seen in [25], p. 88. In most literature, the self-allied cone is called normal cone). A super efficient solution is a Henig efficient solution, if C has a base. A Henig efficient solution is an efficient solution. An efficient solution is a weak efficient solution, if int $C \neq \emptyset$.

Thus, a strong solution of VEP is an ideal solution, better than other solutions. Hence it is important to study the existence of the strong solution and the properties of the strong solutions set.

As we know, there are very few papers that dealt with the existence of the strong solution for VEP. Following the suggestions by Fan (see [26]), Martellotti and Salvadori [6] studied the existence of the strong solution for VEP and the strong minimax theorem problem *for function taking valued* in a Riesz space without a linear structure. Ansari et al. [2] presented an existence theorem for the strong vector equilibrium problem by using Kakutani fixed point theorem. Fu [7] established an existence theorem for a generalized strong vector quasi-equilibrium problem by using Kakutani–Fan–Glicksberg fixed point theorem. It should be mentioned that these methods required different hypotheses.

In this paper, we will study the existence of the strong solution, discuss the arc-wise connectedness and the closedness of the strong solution set, and give a necessary and sufficient condition for the strong solution for the vector equilibrium problems.

2 Existence of the strong solution

Fan [27] and Stoer and Witzgall [28] studied the minimax theorem for real valued convex–concave-like function. By using the separation theorem for convex sets, Jeyakumar [29] established the minimax theorem for real valued ε-concave–convex-like function; Gwinner and Oettli [30] obtained the inf-sup theorem for real valued convex-like-almost concave-like function; Gong et al. [5] obtained an existence theorem of super efficient solution to the VEP for almost concave-like convex-like vector valued function. In this section, we give an existence theorem of the strong solution to the VEP for a function which is finite concave-like in its first variable and downward directed in its second variable by using the separation theorem for convex sets.

Now we recall the some definitions.

Definition 2.1 ([31]) Let A be a subset of X. A mapping $g: A \to Z$ is called C-lower (C-upper) semicontinuous at $x_0 \in A$ if for any symmetric neighborhood V of 0 in Z, there is a neighborhood $U(x_0)$ of x_0 in X such that

$$g(x) \in g(x_0) + V + C \quad \text{for all } x \in U(x_0) \cap A$$

$$(g(x) \in g(x_0) + V - C \quad \text{for all } x \in U(x_0) \cap A).$$

g is called C-lower (C-upper) semicontinuous on A if it is C-lower (C-upper) semicontinuous at every $x \in A$.

Definition 2.2 ([31]) A mapping $g: A \to Z$ is called lower (upper) semicontinuous on A if for any $z \in Z$, the set

$$L(z) = \{x \in A: g(x) \le z\} \quad (L'(z) = \{x \in A: g(x) \ge z\})$$

is closed in A.

Lemma 2.1 *If $g: A \to Z$ is C-lower (C-upper) semicontinuous on A, then g is lower (upper) semicontinuous on A.*

Proof Suppose that g is C-lower semicontinuous on A. We will show that for any $z \in Z$, $L(z)$ is a closed subset of$A$. If this is not true, then there exists $z \in Z$ such that $L(z)$ is not a closed subset of A, which means that there exists a net $\{x_\alpha : \alpha \in I\} \subset \{x \in A: g(x) \le z\}$, $\lim x_\alpha = x_0 \in A$, $x_0 \notin L(z)$. Then $z - g(x_0) \notin C$. Since C is a closed convex pointed cone, there exists a symmetric neighborhood V of 0 in Z such that

$$(z - (g(x_0) + V + C)) \cap C = \emptyset. \tag{2.1}$$

Since g is C-lower semicontinuous at x_0, there exists a neighborhood $U(x_0)$ of x_0 in X such that

$$g(x) \in g(x_0) + V + C \quad \text{for all } x \in U(x_0) \cap A.$$

Since $\lim x_\alpha = x_0$, there exists α_0, such that $x_\alpha \in U(x_0) \cap A$, for all $\alpha \geq \alpha_0$. Hence

$$g(x_\alpha) \in g(x_0) + V + C.$$

By (2.1),

$$z - g(x_\alpha) \notin C \quad \text{for all } \alpha \geq \alpha_0. \tag{2.2}$$

On the other hand, by $\{x_\alpha : \alpha \in I\} \subset \{x \in A: g(x) \leq z\}$, we have $z - g(x_\alpha) \in C$. This contradicts (2.2). Hence $L(z)$ is closed. Therefore, g is lower semicontinuous on A.

Similarly, we can show that if g is C-upper semicontinuous on A, then g is upper semicontinuous on A.

Remark 2.1 A mapping $g: A \to Z$ which is lower (upper) semicontinuous at $x_0 \in A$ does not need to be C-lower (C-upper) semicontinuous at $x_0 \in A$. The counterexample can be seen in [31, pp. 22–23].

We introduce the following concepts.

Definition 2.3 Let X be a real Hausdorff topological vector space, Z a locally convex topological vector space, $C \subset Z$ a closed convex pointed cone, and A a nonempty subset of$X$. A bifunction $f: A \times A \to Z$ is said to be finite concave-like in its first variable if, for any finite set $A_0 \subset A$, and for any $y_1, y_2 \in A$ and $t \in [0, 1]$, there exists $y_3 \in A$ such that

$$tf(y_1, x) + (1-t)f(y_2, x) \leq f(y_3, x) \quad \text{for all } x \in A_0.$$

Definition 2.4 A bifunction $f: A \times A \to Z$ is said to be downward directed in its second variable if for any $x_1, x_2 \in A$, there exists $x_3 \in A$ such that

$$f(y, x_3) \leq f(y, x_1), \quad f(y, x_3) \leq f(y, x_2) \quad \text{for all } y \in A.$$

If Z is a Riesz space, the above concept was used by Martellotti and Salvadori in [6]. If $f: A \times A \to Z$ is C-lower semicontinuous and proper quasiconvex in its second variable, then f is downward directed in its second variable. The concept of proper quasiconvex mapping can be seen in [32].

Theorem 2.1 *Let A be a nonempty weakly compact subset of $X, C \subset Z$ a closed convex pointed cone. Assume that $f: A \times A \to Z$ is finite concave-like in its first variable and downward directed in its second variable. If for each $x \in A$, $f(\cdot, x)$ is C-weakly upper semicontinuous on A and for any $x \in A$, $f(x, x) \geq 0$, then $V_s(A, f) \neq \emptyset$.*

Proof Define a set-valued mapping $G: A \to 2^A$ by

$$G(x) = \{y \in A: f(y, x) \geq 0\}, x \in A.$$

By assumption, we have $G(x) \neq \emptyset$ for any $x \in A$. Since for each $x \in A$, $f(\cdot, x)$ is C-weakly upper semicontinuous on A, by Lemma 2.1, $G(x)$ is a weakly closed subset of A. It is clear that $V_s(A, f) \neq \emptyset$ is equivalent to $\bigcap\{G(x): x \in A\} \neq \emptyset$. Thus we need to show that $\bigcap\{G(x): x \in A\} \neq \emptyset$. Since A is weakly compact, we need to show $\bigcap_{i-1}^n G(x_i) \neq \emptyset$ for any arbitrary chosen $x_1, \ldots, x_n$ in A. If this is not true, then there exists a set $A_0 = \{x_1, \ldots, x_n\} \subset A$ with $\bigcap_{i-1}^n G(x_i) = \emptyset$. Hence for any $y \in A$, there exists $x_i \in A_0$ such that $y \notin G(x_i)$. This means that $f(y, x_i) \notin C$. Since C is a closed convex pointed cone, there exists a neighborhood V_{y,x_i} of zero such that

$$(f(y, x_i) + V_{y,x_i} - C) \cap C = \emptyset. \tag{2.3}$$

There exists an open convex symmetric neighborhood U_{y,x_i} of zero such that

$$U_{y,x_i} + U_{y,x_i} \subset V_{y,x_i}.$$

Since $f(\cdot, x_i)$ is C-weakly upper semicontinuous at y, there exists a weakly open neighborhood $U(y)$ of y in X such that

$$f(y', x_i) \in f(y, x_i) + U_{y,x_i} - C \quad \text{for all } y' \in U(y) \cap A. \tag{2.4}$$

By (2.3) and (2.4), we can get

$$(f(y', x_i) + U_{y,x_i} - C) \cap C = \emptyset \quad \text{for all } y' \in U(y) \cap A. \tag{2.5}$$

Thus, for each $y \in A$, there exist $x_i \in A_0$ and neighborhood $U(y)$ of y, and an open convex symmetric neighborhood U_{y,x_i} of zero such that (2.5) holds.

Thus $A \subset \bigcup\{U(y) : y \in A\}$. Since A is weakly compact, there exists $U(y_1), \ldots, U(y_m)$ with

$$A \subset \bigcup\{U(y_j) : j = 1, \ldots, m\}. \tag{2.6}$$

We set

$$V = \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} W_{y_j,x_i},$$

where for each y_j, and $i = 1, \ldots, n$, if $f(y_j, x_i) \notin C$, we set $W_{y_j,x_i} = U_{y_j,x_i}$; otherwise, we set $W_{y_j,x_i} = X$. It follows from (2.5) and (2.6) that for any $y \in A$, there exists $x_i \in A_0$ such that

$$(f(y, x_i) + V - C) \cap C = \emptyset. \tag{2.7}$$

Set $h(y) = (f(y, x_1) + V - C,\ f(y, x_2) + V - C, \ldots, f(y, x_n) + V - C), \quad y \in A.$ We show that

$$h(A) = \bigcup_{y\in A} h(y) = \bigcup_{y\in A} (f(y, x_1) + V - C,\ f(y, x_2) + V - C, \ldots, f(y, x_n) + V - C)$$

is a convex subset of $Z \times Z \times \cdots \times Z$. Take $(z_1, \ldots, z_n), (z'_1, \ldots, z'_n) \in h(A),\ t \in [0, 1]$. Then, there exist $y_1, y_2 \in A$ and $v_1, \ldots, v_n,\ v'_1, \ldots, v'_n \in V$, $c_1, \ldots, c_n,\ c'_1, \ldots, c'_n \in C$ such that

$$(z_1, \ldots, z_n) = (f(y_1, x_1) + v_1 - c_1,\ f(y_1, x_2) + v_2 - c_2, \ldots, f(y_1, x_n) + v_n - c_n)$$

and

$$\begin{aligned}(z'_1, \ldots, z'_n) &= (f(y_2, x_1) + v'_1 - c'_1,\ f(y_2, x_2) + v'_2 - c'_2, \ldots, f(y_2, x_n) + v'_n - c'_n).\\ &\quad t(z_1, \ldots, z_n) + (1-t)(z'_1, \ldots, z'_n)\\ &= (tf(y_1, x_1) + (1-t)f(y_2, x_1) + tv_1 + (1-t)v'_1 - (tc_1 + (1-t)c'_1), \ldots,\\ &\quad tf(y_1, x_n) + (1-t)f(y_2, x_n) + tv_n + (1-t)v'_n - (tc_n + (1-t)c'_n).\end{aligned} \tag{2.8}$$

Since $f: A \times A \to Z$ is finite concave-like in its first variable, for above $A_0 = \{x_1, \ldots, x_n\}$ and $y_1, y_2 \in A$ and $t \in [0, 1]$, there exists $y_3 \in A$ such that

$$tf(y_1, x) + (1-t) f(y_2, x) \leq f(y_3, x) \quad \text{for all } x \in \{x_1, \ldots, x_n\} = A_0.$$

Thus, there exists $\bar{c_i} \in C$ such that

$$tf(y_1, x_i) + (1-t) f(y_2, x_i) = f(y_3, x_i) - \bar{c_i} \quad \text{for all } i = 1, 2, \ldots, n. \tag{2.9}$$

From (2.8) and (2.9), we get

$$\begin{aligned} & t(z_1, \ldots, z_n) + (1-t)(z'_1, \ldots, z'_n) \\ & \in (f(y_3, x_1) + V - C,\ f(y_3, x_2) + V - C, \ldots,\ f(y_3, x_n) + V - C) = h(y_3) \\ & \subset \bigcup_{y \in A} h(y) = h(A). \end{aligned}$$

Since V is an open set, $h(A)$ is an open subset of $Z \times Z \times \cdots \times Z$. It follows from (2.7) that

$$h(A) \bigcap (C \times C \times \cdots \times C) = \emptyset.$$

By the separation theorem of convex sets (see [33]), there exists $0 \neq (\varphi_1, \ldots, \varphi_n) \in (Z \times Z \times \cdots \times Z)^* = Z^* \times Z^* \times \cdots \times Z^*$ such that

$$\begin{aligned} & \varphi_1(f(y, x_1) + v_1 - c_1) + \varphi_2(f(y, x_2) + v_2 - c_2) + \cdots + \varphi_n(f(y, x_n) + v_n - c_n) \\ & \quad \leq \varphi_1(c'_1) + \varphi_2(c'_2) + \cdots + + \varphi_n(c'_n) \quad \text{for all} \quad c_i, c'_i \in C, v_i \in V, i = 1, \ldots, n, y \in A. \end{aligned}$$

We can see $\varphi_i \in C^*$, for all $i = 1, \ldots, n$, and

$$\begin{aligned} & \varphi_1(f(y, x_1) + v_1) + \varphi_2(f(y, x_2) + v_2) + \cdots + \varphi_n(f(y, x_n) + v_n) \\ & = \varphi_1(f(y, x_1)) + \varphi_1(v_1) + \varphi_2(f(y, x_2)) + \varphi_2(v_2) + \cdots + \varphi_n(f(y, x_n)) + \varphi_n(v_n) \leq 0 \end{aligned}$$

for all $y \in A$, $v_1, \ldots, v_n \in V$. Since $(\varphi_1, \ldots, \varphi_n) \neq 0$ and V is symmetric, there exist $\delta_1 \geq 0, \ldots, \delta_n \geq 0$ with $\delta = \delta_1 + \cdots + \delta_n > 0$ such that

$$\varphi_1(f(y, x_1)) + \varphi_2(f(y, x_2)) + \cdots + \varphi_n(f(y, x_n)) \leq -(\delta_1 + \cdots + \delta_n) \quad \text{for all } y \in A.$$

That is

$$(\varphi_1, \ldots, \varphi_n)(f(y, x_1), (f(y, x_2), \ldots, f(y, x_n)) \leq -\delta \quad \text{for all } y \in A. \tag{2.10}$$

Since f is downward directed in its second variable, by induction, we can see that there exists $x \in A$ such that

$$f(y, x) \leq f(y, x_i) \quad \text{for all } i = 1, \ldots, n, \text{ and for all } y \in A. \tag{2.11}$$

It follows from (2.11) and (2.10), and $\varphi_i \in C^*$, $i = 1, \ldots, n$,

$$(\varphi_1, \ldots, \varphi_n)(f(y, x), (f(y, x), \ldots, f(y, x)) \leq -\delta \quad \text{for all } y \in A.$$

Taking $y = x$, we have

$$(\varphi_1, \ldots, \varphi_n)((f(x, x), (f(x, x), \ldots, f(x, x)) \leq -\delta < 0.$$

However, by assumption, $f(x, x) \geq 0$ and $\varphi_i \in C^*$, we have

$$(\varphi_1, \ldots, \varphi_n)((f(x, x), (f(x, x), \ldots, f(x, x)) \geq 0.$$

This is a contradiction. Thus,

$$\bigcap\{G(x) : x \in A\} \neq \emptyset.$$

There is $x_0 \in \bigcap\{G(x) : x \in A\}$. This means that

$$f(x_0, x) \geq 0 \quad \text{for all } x \in A.$$

By the definition, we have $x_0 \in V_s(A, f)$.

Example 2.1 Let $X = R$, $A = [-1, 1] \subset X$, $Z = R^2$, and $C = R^2_+ = \{(x, y) : x \geq 0, y \geq 0\}$. Let $f: A \times A \to R^2$ be defined by $f(y, x) = (y - x, (y - x)^3)$, $(y, x) \in [-1, 1] \times [-1, 1]$.

It is easy to see that the condition of Theorem 2.1 is satisfied. We have $1 \in V_s(A, f)$.

Definition 2.5 Let $A \subset X$. A bifunction $f: A \times A \to Z$ is called to be arc-concave-like in its first variable if, for any $y_1, y_2 \in A$, there exists an arc $I_{y_1,y_2}(t) \subset A$ such that, for each $t \in [0, 1]$, the following property holds:

$$tf(y_1, x) + (1 - t)f(y_2, x) \leq f(I_{y_1,y_2}(t), x) \quad \text{for all } x \in A,$$

where $I_{y_1,y_2} : [0, 1] \to A$ is a continuous map with $I_{y_1,y_2}(0) = y_1$, $I_{y_1,y_2}(1) = y_2$.

Remark 2.2 It is easy to see that, if f is arc-concave-like in its first variable, then, f is finite concave-like in its first variable.

Theorem 2.2 *Let A be a nonempty weakly compact subset of X, $C \subset Z$ a closed convex pointed cone. Suppose that mapping $f: A \times A \to Z$ is arc-concave-like in its first variable and downward directed in its second variable. Furthermore, if for any $x \in A$, $f(\cdot, x)$ is C-weakly upper semicontinuous on A and for any $x \in A$, $f(x, x) \geq 0$, then, $V_s(A, f)$ is a weakly closed arc-wise connected set.*

Proof By Remark 2.2 and Theorem 2.1, we know that $V_s(A, f) \neq \emptyset$. We show that $V_s(A, f)$ is an arc-wise connected set. Consider $y_1, y_2 \in V_s(A, f)$, then, for $i = 1, 2$,

$$f(y_i, x) \geq 0 \quad \text{for all } x \in A.$$

For any $t \in [0, 1]$, we have

$$0 \leq tf(y_1, x) + (1 - t)f(y_2, x) \leq f(I_{y_1,y_2}(t), x) \quad \text{for all } x \in A,$$

since $f(y, x)$ is arc-concave-like with respect to its first variable. Thus, we have $I_{y_1,y_2}(t) \in V_s(A, f)$, for all $t \in [0, 1]$. Hence, $V_s(A, f)$ is an arc-wise connected set.

Now we show that $V_s(A, f)$ is a weakly closed subset of X. Let a net $\{x_\alpha : \alpha \in I\} \subset V_s(A, f)$ be given such that $\{x_\alpha : \alpha \in I\}$ weakly converges to $\bar{x}$. Since $\{x_\alpha : \alpha \in I\} \subset A$ and A is weakly compact, $\bar{x} \in A$. For any $y \in A$, we have

$$\{x_\alpha : \alpha \in I\} \subset \{x \in A : f(x, y) \geq 0\}.$$

Since $f(\cdot, y)$ is C-weakly upper semicontinuous on A, by Lemma 2.1, $\{x \in A : f(x, y) \geq 0\}$ is a weakly closed subset in A. Thus, we have $\bar{x} \in \{x \in A : f(x, y) \geq 0\}$. By the arbitrary of $y \in A$, we have

$$f(\bar{x}, y) \geq 0 \quad \text{for all } y \in A.$$

This means that $\bar{x} \in V_s(A, f)$. Thus, $V_s(A, f)$ is weakly closed.

Example 2.2 Let $X = R$, $A = [-1, 1] \subset X$, $Z = R^2$, and $C = R^2_+$. Let $f: A \times A \to R^2$ be defined by $f(x, y) = (1 + x - y, 1 + x - y)$, $(x, y) \in A \times A$.

It is easy to see that the condition of Theorem 2.2 is satisfied. It is clear that $[0, 1] = V_s(A, f)$ is a closed convex set.

3 The application

Now let $g: A \to Z$ be a mapping. We consider the vector optimization problem

$$\text{(VOP)} \qquad \min_{x \in A} g(x).$$

Definition 3.1 $x_0 \in A$ is called a strong solution for (VOP) if

$$g(x) \geq g(x_0) \quad \text{for all } x \in A.$$

Aubin and Ekeland [34], Jahn and Rauh [35], and Gong et al. [36] have studied the optimization condition for the strong solution of vector optimization problem. In this section, we will apply Theorem 2.1 to get the existence theorem of strong solution for (VOP).

Definition 3.2 Let $g: A \to Z$ be a mapping. g is said to be convex-like, if for any $x_1, x_2 \in A$, $t \in [0, 1]$, there exists $x_3 \in A$, such that

$$g(x_3) \leq tg(x_1) + (1 - t)g(x_2).$$

Definition 3.3 Let $g: A \to Z$ be a mapping. g is said to be downward directed if for any $x_1, x_2 \in A$, there exists $x_3 \in A$, such that

$$g(x_3) \leq g(x_1), g(x_3) \leq g(x_2).$$

Proposition 3.1 *Let A be a nonempty weakly compact subset of X, $C \subset Z$ a closed convex pointed cone. Assume that $g: A \to Z$ is a convex-like, downward directed, and C-weakly lower semicontinuous mapping on A. Then there exists some $x_0 \in A$, which is a strong solution for (VOP).*

Proof Set $f(y, x) = g(x) - g(y)$, $(y, x) \in A \times A$. By the assumption, we know that $f: A \times A \to Z$ satisfies the condition that Theorem 2.1. By Theorem 2.1, there exists some $x_0 \in A$ such that

$$f(x_0, x) \geq 0 \quad \text{for all } x \in A.$$

Thus we have

$$g(x) \geq g(x_0) \quad \text{for all } x \in A.$$

This means that x_0 is a strong solution for (VOP). □

4 The necessary and sufficient condition

In this section, we give a necessary and sufficient condition for the strong solution to the VEP, which reveals the relation between the strong solution and the efficient solution to the VEP.

Definition 4.1 ([25] p. ix) Let D be a nonempty subset of Z. D is called downward directed, if for every $d_1, d_2 \in D$, there exists a $d \in D$ such that $d \leq d_{1,} d \leq d_2$.

Theorem 4.1 *Let A be a nonempty compact subset of X, $C \subset Z$ a closed convex point cone. Let $f : A \times A \to Z$ be a given mapping. Assume that for each $x \in A, f(x, \cdot)$ is lower semicontinuous on A, and $f(x, x) = 0$ for all $x \in A$. Then, $x_0 \in V_s(A, f)$ if and only if $x_0 \in V(A, f)$ and $f(x_0, A)$ is downward directed.*

Proof Let $x_0 \in V_s(A, f)$. It is clear that $x_0 \in V(A, f)$, since C is a pointed cone. For any $y_1, y_2 \in A$, by $x_0 \in V_s(A, f)$, we have

$$0 \leq f(x_0, y_1), 0 \leq f(x_0, y_2).$$

By assumption,

$$0 = f(x_0, x_0) \leq f(x_0, y_1), \quad 0 = f(x_0, x_0) \leq f(x_0, y_2).$$

Thus, $f(x_0, A)$ is downward directed.

Conversely, let $x_0 \in V(A, f)$ and $f(x_0, A)$ be downward directed. For every $y \in A$, let $L(y) = \{x \in A\colon f(x_0, x) \leq f(x_0, y)\}$. Since $f(x_0, \cdot)$ is lower semicontinuous on A, $L(y)$ is closed in A. It is clear that $y \in L(y), L(y) \neq \emptyset$. For $y_1, \dots y_n \in A$, since $f(x_0, A)$ is downward directed, by induction, we can see that there exists $y \in A$ such that

$$f(x_0, y) \leq f(x_0, y_i), \quad i = 1, \dots, n.$$

We have

$$y \in \bigcap_{i=1}^{n} L(y_i).$$

Since A is compact, $\bigcap_{y \in A} L(y) \neq \emptyset$. Hence, there exists $y_0 \in \bigcap_{y \in A} L(y)$.
We have

$$f(x_0, y_0) \leq f(x_0, y) \quad \text{for all } y \in A. \tag{4.1}$$

Hence,

$$f(x_0, y_0) \leq f(x_0, x_0) = 0.$$

We have

$$f(x_0, y_0) \in -C.$$

If $f(x_0, y_0) \neq 0$, then

$$f(x_0, y_0) \in -C \backslash \{0\}. \tag{4.2}$$

Since $x_0 \in V(A, f)$, we have

$$f(x_0, y) \notin -C \backslash \{0\} \quad \text{for all } y \in A.$$

This contradicts (4.2). Hence, we have $f(x_0, y_0) = 0$. By (4.1), we have

$$0 \leq f(x_0, y) \quad \text{for all } y \in A.$$

This means that $x_0 \in V_s(A, f)$.

Acknowledgements The author is grateful to the referees for their valued comments. This work was partially supported by the Natural Science Foundation of China and the Natural Science Foundation of Jiangxi Province, China.

References

1. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Student **63**, 123–145 (1994)
2. Ansari, Q.H., Oettli, W., Schläger, D.: A generalization of vectorial equilibria. In: Operations Research and Its Applications, Second International Symposium, ISORA'96 Guilin, P.R. China, December 11–14, 1996 Proceedings, World Publishing Corporation, 181–185 (1996)
3. Giannessi, F., Mastroeni, G., Pellegrini, L.: On the theory of vector optimization and variational inequalities. Image space analysis and separation. In: Giannessi (ed.) Vector Variational Inequalities and Vector Equilibria, Mathematical Theories, pp. 153–215. Kluwer Academic Publishers (2000)
4. Gong, X.H.: Efficiency and Henig efficiency for vector equilibrium problems. J. Optim. Theor. Appl **108**, 139–154 (2001)
5. Gong, X.H., Fu, W.T., Liu, W.: Super efficiency for a vector equilibrium in locally convex topological vector spaces. In: Giannessi (ed.) Vector Variational Inequalities and Vector Equilibria, Mathematical Theories, pp. 233–252. Kluwer Academic Publishers (2000)
6. Martellotti, A., Salvadori, A.: Inequality systems and minimax results without linear structure. J. Math. Anal. Appl. **148**, 79–86 (1990)
7. Fu, J.Y.: Generalized vector quasi-equilibrium problem. Math. Methods Operation Res. **52**, 57–64 (2000)
8. Chen, G.Y., Yang, X.Q.: Vector complimentarity problem and its equivalence with weak minimal element in ordered spaces. J. Math. Anal. Appl. **153**, 136–158 (1990)
9. Chen, G.Y.: Existence of solution for a vector variational inequality: an extension of the Hartman-Stampacchia theorem. J. Optim. Theor. Appl. **74**, 445–456 (1992)
10. Yang, X.Q.: Vector varitional inequality and its duality. Nonlinear Anal. Theor. Methods Appl. **21**, 869–877 (1993)
11. Siddiqi, A.H., Ansari, Q.H., Khaliq, A.: On vector variational inequalities. J. Optim. Theor. Appl. **84** 171–180 (1995)
12. Chen, G.Y., Li, S.J.: Existence of solution for a generalized vector quasivariational Inequality. J. Optim. Theor. Appl. **90**, 321–334 (1996)
13. Yu, S.J., Yao, J.C.: On vector variational inequalities. J. Optim. Theor. Appl. **89**, 749–769 (1996)
14. Bianch, M., Hadjisavvas, N., Schaible, S.: Vector equilibrium problems with generalized monotone bifunctions. J. Optim. Theor. Appl. **92**, 527–542 (1997)
15. Konnov, I.V., Yao, J.C.: Existence of solutions for generalized vector equilibrium problems. J. Math. Anal. Appl. **233**, 328–335 (1999)
16. Lee, G.M., Lee, B.S., Chang, S.S.: On vector quasivariational inequalities. J. Math. Anal. Appl. **203**, 626–638 (1996)
17. Giannessi, F. (ed.): Vector Variational Inequalities and Vector Eqilibria. Mathematical Theories. Kluwer Academic Publishers (2000)
18. Ansari, Q.H., Schaible, S., Yao, J.C.: System of vector equilibrium problems and its applications. J. Optim. Theor. Appl. **107**, 547–557 (2000)
19. Mastroeni, G., Rapcsák, T.: On convex generalized systems. J. Optim. Theor. Appl. **104**, 605–627 (2000)
20. Bianchi, M., Pini, R.: A result on localization of equilibria. J. Optim. Theor. Appl. **115**, 335–343 (2002)
21. Fabián F.B., Frnando, F.B.: Vector equilibrium problems under asymptotic analysis. J. Global Optim. **26**, 141–166 (2003)
22. Lin, L.J., Ansari, Q.H., Wu, J.Y.: Geometric properties and coincidence theorems with applications to generalized vector equilibrium problems. J. Optim. Theor. Appl. **117**, 121–137 (2003)
23. Li, J., Huang, N.J., Kim J.K.: On implicit vector equilibrium problems. J. Math. Anal. Appl. **283**, 501–512 (2003)
24. Fu, J.Y.: Symmetric vector quasi-equilibrium problems. J. Math. Anal. Appl. **285**, 708–713 (2003)
25. Jameson, G.: Ordered Linear Spaces, Lecture Notes in Math. Springer-Verlag, Berlin, 141p. (1970)
26. Martellotti, A., Salvadori, A.: A minimax theorem for functions taking values in a Riesz space. J. Math. Anal. Appl. **133**, 1–13 (1988)
27. Fan, K.: Minimax Theorem. National Academy of Sciences, Washington, DC, Proceedings USA **39**, 42–47 (1953)
28. Stoer, J., Witzgall, C.: Convexity and Optimization in Finite Dimensions: I. Springer-Verlag, New York (1970)
29. Jeyakumar, V.: A generalization of a minimax theorem of Fan via a theorem of the alternative. J. Optim. Theor. Appl. **48**, 525–533 (1986)
30. Gwinner, J., Oettli, W.: Theorems of the alternative and duality for inf-sup problems. Math. Operations Res. **19**, 238–256 (1994)
31. Luc, D.T.: Theory of Vector Optimization, Lecture Notes in Economics and Mathematical Systems. Spring-Verlag (1989)

32. Ferro, F.: A minimax theorem for vector-valued functions. J. Optim. Theor. Appl. **60**, 19–31 (1989)
33. Jahn, J.: Mathematical Vector Optimization in Partially Ordered Linear Spaces. Verlag Peter Lang, New York (1986)
34. Aubin, J.P., Ekeland, I.: Applied Nonlinear Analysis. John Wiley, New York (1984)
35. Jahn, J., Rauh, R.: Contingent epiderivatives and set-valued optimization. Math. Methods Operation Res. **46**, 193–211 (1997)
36. Gong, X.H., Dong, H.B., Wang. S.Y.: Optimality conditions for proper efficient solutions of vector set-valued optimization. J. Math. Anal. Appl. **284**, 332–350 (2003)